Topology Determination of Solution Sets in Subdivision Multivariate Solvers

Yonathan Mizrahi, Department of Computer Science, Technion

Consider the problem of solving a system of nonlinear algebraic equations:

$$F(\bar{x}) = \bar{0}$$

When $F: \mathbb{R}^n \longrightarrow \mathbb{R}^k$ with $n \geq k$.

Under suitable smoothness and regularity assumptions – the solution set:

$$S = \{\bar{x} \in \mathbb{R}^n : F(\bar{x}) = \bar{0} \in \mathbb{R}^k\}$$

is a sub-manifold of $\mathbb{R}^n$ of dimension $n - k$.

Bezier/B-spline Representation

We represent each of the scalar constraints in the Bezier or B-spline form:

$$F_i(\bar{x}) = \sum_{j=0}^{m} \mathcal{B}_j(\bar{x}) P_j$$

Where:

$\{P_j\}_{j=0,\dots,m}$ are constant scalar coefficients (control points).

$\{\mathcal{B}_j\}_{j=0,\dots,m}$ are (basis) blending functions.

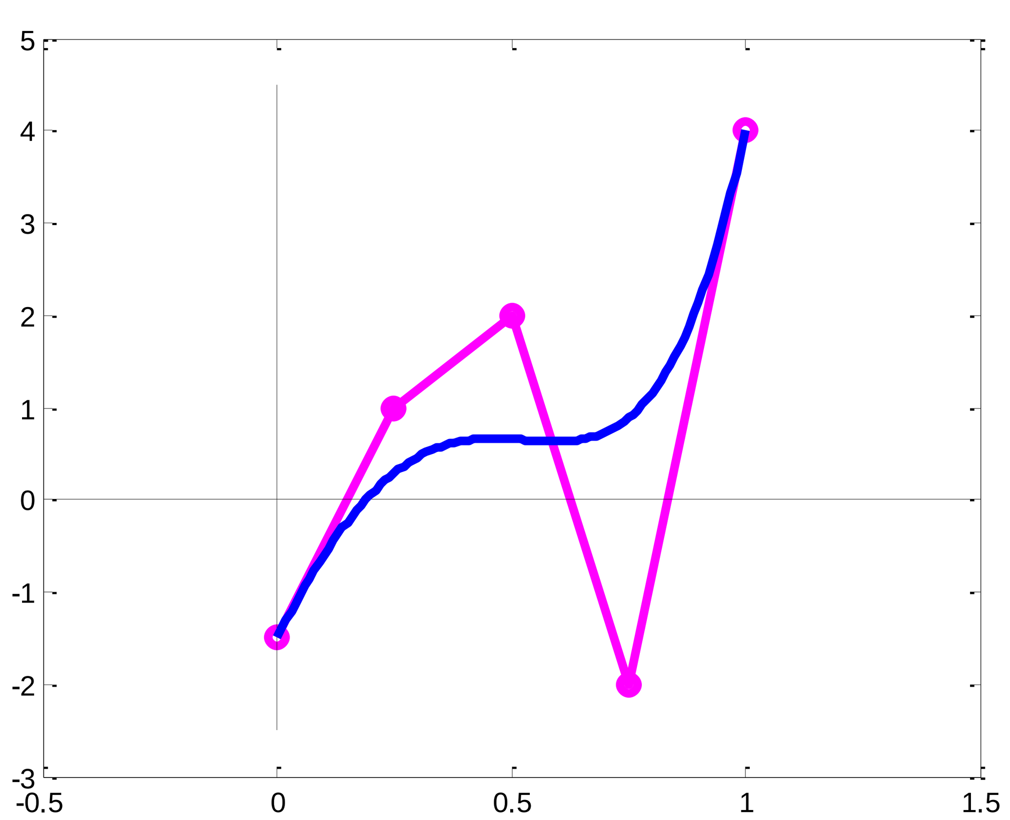
Example – Bezier Curves

The case $n = k = 1$ is the problem of finding the zeros (or roots) of a single curve.

Bezier curves are given by:

$$f(x) = \sum_{j=0}^{m} \mathcal{B}_j(x) P_j = \sum_{j=0}^{m} \binom{m}{j} (1-x)^{m-j} x^j P_j$$

When the blending functions are the ***Bernstein basis polynomials (of degree m).***



The Convex Hull Property and the Subdivision Approach

Bezier/B-spline multivariates are contained in the convex hull of their control points [1].

We seek solutions of $F(\bar{x}) = \bar{0}$ in some domain $D \subset \mathbb{R}^n$.

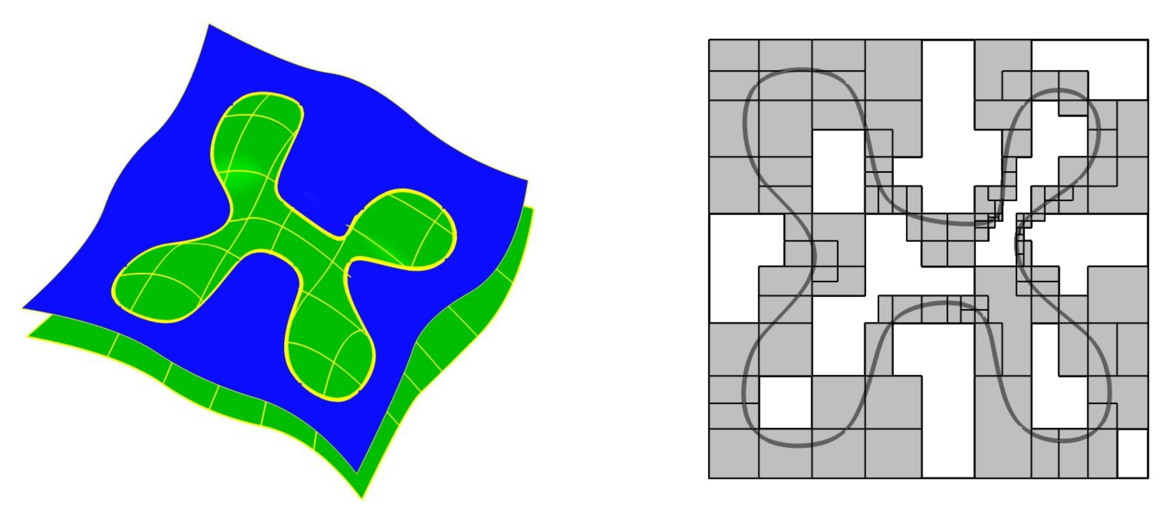
Subdivision of D provides new control points in each subdomain – closer (in fact converging) to the multivariate.

Subdomains in which control points do not change sign can be purged away [2].

The subdivision process is followed by numeric improvement of the approximated solution set.

Example - SSI

The case $n = 3, k = 2$ is the Surface-Surface Intersection problem.



Two B-splines surfaces with closed curve intersection (left). Preimage in the parameter space of the **blue** surface.

The Problem

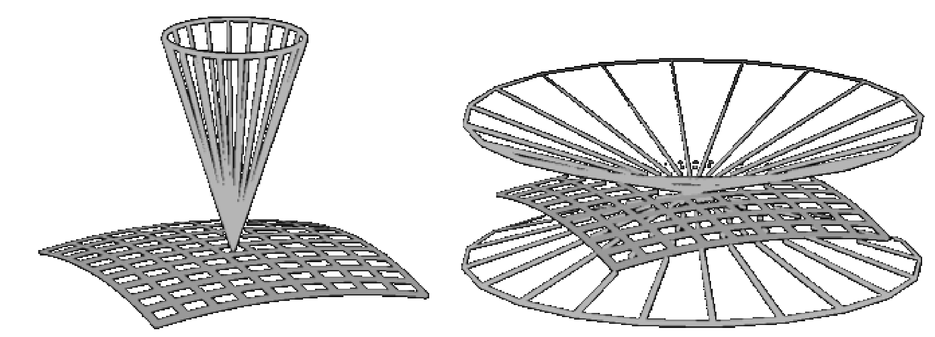
Subdivision gets computationally expensive ***as the dimension of the problem increases.***

We seek a subdivision algorithm with topologically guaranteed termination criteria.

The topology of the solution set within each remaining subdomain is ***required to be figured out and be "simple" enough – prior to the numeric improvement!***

The Zero Dimensional Solution Set

In fully determined systems - subdivision is terminated ***when roots are isolated .*** Consider the normal and the tangent hyper-cones of each hyper-surface of the system:



<u>*Theorem*</u>: Given n implicit hyper-surfaces in $\mathbb{R}^n$, there exists at most one common solution if the intersection of their tangent hyper-cones is $\{\bar{0}\}$.

(Hanniel & Elber, [3]).

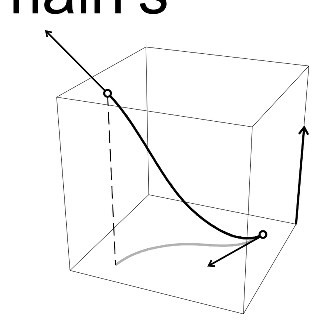
The Univariate Solution Set

The zero set of $F: \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ is a curve in $\mathbb{R}^n$.

Subdivision is terminated when only a ***single component and no loops*** exist in each subdomain (Barton, Hanniel & Elber, [4]).

The No Loop Test (NLT) inspects the image in S^{n-1} of the unit tangents to the curve.

The Single Component Test (SCT) checks for exactly two intersections with the domain's boundary.



Current Research

We seek subdivision algorithms with topologically guaranteed termination criteria for higher dimensional solution spaces – currently for 2-manifolds ($F: \mathbb{R}^n \longrightarrow \mathbb{R}^{n-2}$).

References:

[1] E. Cohen, R.F. Riesenfeld and G. Elber, ***Geometric Modeling with Splines, An Introduction.*** A K Peters, LTD. 2001.

[2] G. Elber and M. S. Kim, ***Solving Geometric Constraints Using Multivariate Rational Spline Functions.*** Sixth ACM on Solid Modeling and Applications, 2001.

[3] G. Elber and I. Hanniel, ***Subdivision Termination Criteria in Subdivision Multivariate Solvers Using Dual Hyper-planes Representations.*** Lecture Notes in Computer Science. **4077**, 115-128, 2006.

[4] M. Barton, G. Elber and I. Hanniel, ***Topologically Guaranteed Univariate Solutions of Under Constrained Polynomial Systems via No-Loop and Single-Component Tests.*** Symposium on Solid and Physical Modeling, 207-212, 2010.